Information-Theoretic Uncertainty Relation and Random-Phase Entropy

Kyoung Kon Kim and Sang Pyo Kim*
Department of Physics, Kunsan National University, Kunsan 573-701, Korea

Sok Kuh Kang
Korea Ocean Research & Development Institute, Ansan 426-744, Korea
(Dated: January 18, 2010)

Dunkel and Trigger [Phys. Rev. A **71**, 052102 (2005)] show that the Leipnik's joint entropy monotonously increases for the initially maximally classical Gaussian wave packet for a free particle. After expressing the joint entropy of the general Gaussian wave packets for quadratic Hamiltonians as $S(t) = \ln(e/2) + \ln(2\Delta x(t)\Delta p(t)/\hbar)$, we show that a class of general Gaussian wave packets does not warrant the monotonous increase of the joint entropy. We propose that the random-phase entropy with respect to the squeeze angle always monotonously increases even for non-maximally classical states.

PACS numbers: 03.65.Ta, 03.67.-a, 05.30.-d

I. INTRODUCTION

The Leipnik's joint entropy is an important measure of information loss of quantum states. Recently Dunkel and Trigger [1] have shown that the Leipnik's joint entropy monotonously increases for a free Gaussian wave packet and a wave packet in a monochromatic electromagnetic field when it is initially prepared as a maximally classical state. They have further argued that the monotonous increase of the entropy may be an intrinsic property for other types of initial wave packets. However, it was shown in Ref. [2] that the Leipnik's joint entropy for a Gaussian wave packet of a harmonic oscillator oscillates between the maximum and the lower bound, which was also confirmed in Ref. [3].

In the present paper, we study the Leipnik's joint entropy for the general Gaussian wave packets of quadratic Hamiltonians, whose mass and frequency may depend on time, and which may be driven by an external force. The exact Gaussian wave packets involve the initial position and momentum for the centroid, and the initial position and momentum variances, which are equivalent to a squeeze parameter and a squeeze angle [4]. We first express the Leipnik's joint entropy in terms of the uncertainty relation and then investigate the condition for the monotonous increase of the entropy. The joint entropy of general Gaussian wave packets for a harmonic oscillator oscillates between the maximum and the lower bound of the joint entropy, while a class of Gaussian wave packets of a free particle has the joint entropy that first decreases and then monotonously increases. However, the random-phase entropy with respect to the squeeze angle indeed monotonously increases for the general Gaussian wave packets for quadratic Hamiltonian systems, including a free particle, a harmonic oscillator and the Caldirola-Kanai Hamiltonian with an exponentially varying mass [5], regardless of an external driving force.

II. JOINT ENTROPY FOR GAUSSIAN WAVE PACKETS

We consider the quadratic Hamiltonians of the form

$$H(t) = \frac{p^2}{2m(t)} + \frac{m(t)\omega^2(t)}{2}x^2 - f(t)x. \tag{1}$$

As shown in Ref. [6], the most general quadratic Hamiltonian may be canonically transformed into the form (1). A harmonic oscillator has constant m_0 and ω_0 while a free particle is the limit of $\omega = 0$. A charged particle in a monochromatic electric field is described by $f(t) = qE\cos(\omega t)$. The Hamiltonian with time-dependent mass or frequency describes an open system. To find the exact Gaussian wave packet for (1), we adopt the invariant method by Lewis and Riesenfeld [7] (see also Ref. [8]). The Hamiltonian (1) has a pair of linear invariant operators [9–12], which leads to the most general Gaussian wave packet [4]

$$\Psi(x,t) = \left(\frac{1}{\sqrt{2\pi\hbar}u^*(t)}\right)^{1/2} e^{-iS_c(t)/\hbar}e^{ip_c(t)x/\hbar} \times e^{-i\frac{m\dot{u}^*}{2\hbar u^*}(x-x_c(t))^2}. \tag{2}$$

A direct calculation shows that (2) indeed satisfies the time-dependent Schrödinger equation. Here, u is a complex solution to

$$\ddot{u} + \frac{\dot{m}}{m}\dot{u} + \omega^2(t)u = 0, \tag{3}$$

satisfying the Wronskian condition

$$m(u\dot{u}^* - \dot{u}u^*) = i. \tag{4}$$

And, x_c is the classical position satisfying

$$\ddot{x}_c + \frac{\dot{m}}{m}\dot{x}_c + \omega^2(t)u_c = f(t), \tag{5}$$

*Electronic address: sangkim@kunsan.ac.kr

and $p_c(t) = m\dot{x}_c(t)$ the corresponding momentum, and $S_c(t) = \int_0^t dt(p_c^2/2m - m\omega^2 x_c^2/2 + f x_c)$ the classical action. The two variances are given by

$$\begin{aligned} \Delta(x)(t) &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\hbar u^*(t)u(t)}, \\ \Delta(p)(t) &= \sqrt{\langle p^2\rangle - \langle p\rangle^2} = m\sqrt{\hbar \dot{u}^*(t)\dot{u}(t)}. \end{aligned} \tag{6}$$

The Leipnik's joint entropy

$$\begin{aligned} S(t) &= -\int dx|\psi(x,t)|^2 \ln|\psi(x,t)|^2 \\ &\quad -\int dp|\tilde{\psi}(p,t)|^2 \ln|\tilde{\psi}(p,t)|^2 - \ln(2\pi\hbar) \end{aligned} \tag{7}$$

of the Gaussian wave packet (2) is determined by the uncertainty relation as

$$S(t) = \ln\Big(\frac{e}{2}\Big) + \ln\Big(\frac{2\Delta x(t)\Delta p(t))}{\hbar}\Big). \tag{8}$$

Note that the centroid (x_c, p_c) does not give any contribution to the joint entropy. So, the question is how to find the variances for the position and momentum. There is arbitrariness in choosing a complex solution u to Eq. (3) since any linear superposition of u and u^* is also a solution. For instance, the Gaussian wave packet for a free particle with the minimum uncertainty at $t = 0$ is obtained by the solution

$$u_0(t) = \frac{1}{\sqrt{2}}\Big(1 - i\frac{t}{m_0}\Big), \tag{9}$$

and for a harmonic oscillator by

$$u_0(t) = \frac{1}{\sqrt{2m_0\omega_0}} e^{-i\omega_0 t}. \tag{10}$$

Then, the most general solution satisfying Eq. (4) may involve two parameters r and ϑ as

$$u(t) = (\cosh r)u_0(t) + (e^{-i\vartheta}\sinh r)u_0^*(t), \tag{11}$$

where $r \geq 0$ and $2\pi > \vartheta \geq 0$. Here, the r and ϑ have the interpretation as the squeeze parameter and angle [10, 11], which are equivalent to determining two integration constants, $\Delta x(0)$ and $\Delta p(0)$. In fact, the general solution (11) leads to the Bogoliubov transformation between the Fock bases constructed from $u(t)$ and $u_0(t)$:

$$\begin{aligned} a(t) &= (\cosh r)a_0(t) - (e^{i\vartheta}\sinh r)a_0^\dagger(t), \\ a^\dagger(t) &= (\cosh r)a_0^\dagger(t) - (e^{-i\vartheta}\sinh r)a_0(t). \end{aligned} \tag{12}$$

Though the free particle allows a shift of time, which corresponds to choosing a set of r and ϑ, the Hamiltonian (1) does not preserve the time-translational symmetry, so we select the wave packets at $t = 0$ by adjusting r and ϑ.

First, we consider the joint entropy for the free particle with $\omega = 0$. As the external force only governs the centroid of the Gaussian wave packet, the joint entropy

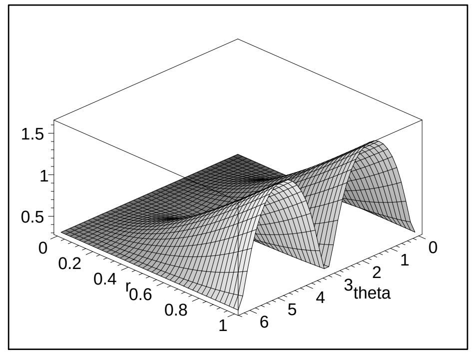

FIG. 1: The entropy change $S(0) - \ln(e/2)$ for the free particle as the function of the squeeze parameter r and the squeeze angle ϑ is drawn in the ranges $1 \geq r \geq 0$ and $2\pi > \vartheta \geq 0$.

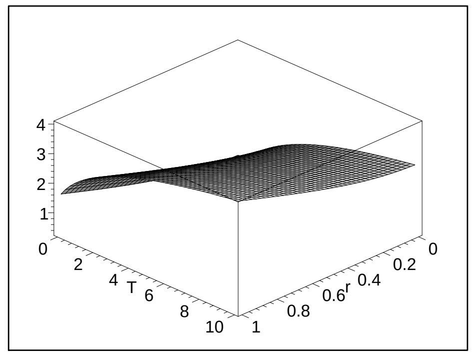

FIG. 2: The entropy change $S(t) - \ln(e/2)$ for the free particle with $\vartheta = \pi/2$ as the function of time $T = t/m_0$ and the squeeze parameter r.

does not depend on the details of the external force. We thus obtain the Leipnik's joint entropy

$$\begin{aligned} S(t) &= \ln\Big(\frac{e}{2}\Big) + \frac{1}{2}\ln\Big(\cos^2\frac{\vartheta}{2} + e^{4r}\sin^2\frac{\vartheta}{2}\Big) + \\ &\frac{1}{2}\ln\Big((\cos\frac{\vartheta}{2} + \frac{t}{m_0}\sin\frac{\vartheta}{2})^2 + e^{-4r}(\sin\frac{\vartheta}{2} - \frac{t}{m_0}\cos\frac{\vartheta}{2})^2\Big). \end{aligned} \tag{13}$$

The initial entropy at $t = 0$

$$S(0) = \ln\Big(\frac{e}{2}\Big) + \frac{1}{2}\ln(1 + \sin^2\vartheta\sinh^2 2r) \tag{14}$$

depends not only on the squeeze parameter r but also the squeeze angle ϑ, which is drawn in Fig. 1. One interesting point is that the minimum entropy may be provided by the zero-squeeze angle ($\vartheta = 0$) even for the non-zero squeeze parameter r. The entropy (13) monotonously increases against time in the range of squeeze angle ($\pi \geq \vartheta \geq 0$) and has the minimum entropy $S(0)$. This is numerically shown in Fig. 2.

However, in the other range ($2\pi > \vartheta > \pi$) the entropy has the lower bound

$$S_{\min} = \ln\Big(\frac{e}{2}\Big), \tag{15}$$

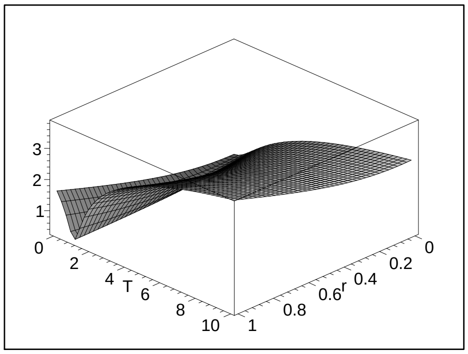

FIG. 3: The entropy change $S(t)-\ln(e/2)$ for the free particle with $\vartheta = 3\pi/2$ as the function of time $T = t/m_0$ and the squeeze parameter r. The entropy reaches the lower bound $\ln(e/2)$ at t_*.

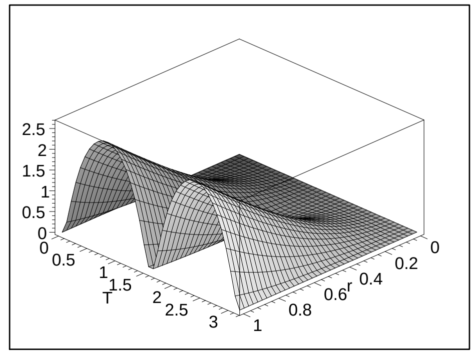

FIG. 4: The entropy change $S(t) - \ln(e/2)$ for the harmonic oscillator is drawn as the function of time $T = \omega_0 t$ and the squeeze parameter r for $\vartheta = 0$.

at $t_* = -(1-e^{-4r})\sin\vartheta/2m(\sin^2(\vartheta/2)+e^{-4r}\cos^2(\vartheta/2))$, which is numerically confirmed in Fig. 3. Thus, the Gaussian wave packets for (1) do not necessarily warrant the monotonous increase of the joint entropy. This counter-example implies that the argument by Dunkel and Triger is only true for the initially maximally classical states with $\Delta x(0)\Delta p(0) = \hbar/2$ [1]. Indeed, the initially maximally classical state is provided either by $\vartheta = 0, \pi$ for any r, leading to the monotonously increasing entropy

$$S(t) = \ln\Big(\frac{e}{2}\Big) + \frac{1}{2}\ln\Big(1 + e^{\pm 4r}\frac{t^2}{m_0^2}\Big), \qquad (16)$$

with the upper/lower sign for $\vartheta = \pi/0$ or by $r = 0$ for any ϑ, leading to another monotonously increasing entropy

$$S(t) = \ln\Big(\frac{e}{2}\Big) + \frac{1}{2}\ln\Big(1 + \frac{t^2}{m_0^2}\Big). \qquad (17)$$

Second, we consider the joint entropy of the general Gaussian wave packets for the harmonic oscillator with m_0 and ω_0. The Leipnik joint entropy is given by

$$S(t) = \ln\Big(\frac{e}{2}\Big) + \frac{1}{2}\ln\Big(1 + \sinh^2(2r)\sin^2(2\omega_0 t - \vartheta)\Big). \qquad (18)$$

As shown in Fig. 4, the joint entropy oscillates between the maximum

$$S_{\max} = \ln\Big(\frac{e}{2}\Big) + \frac{1}{2}\ln\Big(1 + \sinh^2(2r)\Big) \qquad (19)$$

and the lower bound $\ln(e/2)$. This oscillator behavior was first pointed out in Ref. [2].

Thus, we may conclude that the Leipnik's joint entropy for the free particle and the harmonic oscillator does not always monotonously increase for the whole range of the squeeze parameter and squeeze angle. Then, a question may be raised whether one may get an information-theoretic entropy in concord with thermodynamics.

III. RANDOM-PHASE ENTROPY

Now we show that the random-phase entropy always leads to a monotonously increasing entropy for all the Hamiltonians (1) with time-dependent mass or frequency or with the external force. Regarding the squeeze angle ϑ as a random variable for the Gaussian wave packets at t, the random-phase entropy is given by

$$\begin{aligned}\overline{S}(t) \;&=\; \ln\Big(\frac{e}{2}\Big) + \ln\Big(\frac{\cosh(2r)+1}{2}\Big) \\ &\quad + \ln\Big(2m(t)|u_0(t)\dot{u}_0(t)|\Big). \end{aligned} \qquad (20)$$

In deriving Eq. (20), we used the integral $\int_0^{2\pi}\ln(a + b\cos x + c\sin x) = 2\pi\ln((a+\sqrt{a^2-b^2-c^2})/2)$ [13]. The random-phase entropy may be written in terms of the minimal uncertainty as

$$\begin{aligned}\overline{S}(t) \;&=\; \ln\Big(\frac{e}{2}\Big) + \ln\Big(\frac{\cosh(2r)+1}{2}\Big) \\ &\quad + \ln\Big(\frac{2(\Delta x)_{\Psi_0}(\Delta p)_{\Psi_0}}{\hbar}\Big). \end{aligned} \qquad (21)$$

Then, the random-phase entropy is

$$\begin{aligned}\overline{S}(t) \;&=\; \ln\Big(\frac{e}{2}\Big) + \ln\Big(\frac{\cosh(2r)+1}{2}\Big) \\ &\quad + \ln\Big(1 + \frac{t^2}{m_0^2}\Big) \end{aligned} \qquad (22)$$

for the free particle and

$$\overline{S} = \ln\Big(\frac{e}{2}\Big) + \ln\Big(\frac{\cosh(2r)+1}{2}\Big) \qquad (23)$$

for the harmonic oscillator.

As an open system, we consider the time-dependent mass $m(t) = m_0 e^{\gamma t}$, which is well-known as the Caldirola-Kanai Hamiltonian [5]. The solution to Eq. (3) that satisfies (4) and gives the minimum uncertainty is [14]

$$u_0(t) = \frac{e^{-\gamma t/2}}{\sqrt{2m_0\omega}}e^{-i\omega t}, \quad \omega = \sqrt{\omega_0^2 - \Big(\frac{\gamma}{2}\Big)^2}. \qquad (24)$$

Then, the random-phase entropy is

$$\overline{S} = \ln\Big(\frac{e}{2}\Big) + \ln\Big(\frac{\cosh(2r)+1}{2}\Big) + \frac{1}{2}\ln\Big(1+\frac{\gamma^2}{4\omega^2}\Big). \quad (25)$$

Even for $r = 0$, the damping effect contributes to the lower bound of entropy by the amount of the last term, which is related with the generalized minimum uncertainty [14].

For time-dependent Hamiltonians (1), we may find a useful lower and upper bound. Using the identity, $\ln(2m(t)|u_0(t)\dot{u}_0(t)|) = \ln(1+(m(t)d|u_0(t)|^2/dt)^2)/2$, we find the lower bound for the random-phase entropy

$$\overline{S}(t) \geq \ln\Big(\frac{e}{2}\Big) + \ln\Big(\cosh(2r)+1\Big). \quad (26)$$

Similarly, from $\langle H(t)\rangle_{\Psi_0} \geq \hbar\omega(t)m(t)|u_0(t)\dot{u}_0|$, with respect to the Gaussian wave packet Ψ_0 obtained using $u_0(t)$, we find the upper bound

$$\ln\Big(\frac{e}{2}\Big) + \ln\Big(\cosh(2r)+1\Big) + \frac{1}{2}\ln\Big(\frac{\langle H(t)\rangle_{\Psi_0}}{\hbar\omega(t)/2}\Big) \geq \overline{S}(t). \quad (27)$$

IV. CONCLUSION

We have investigated the Leipnik's joint entropy of the general Gaussian wave packets for quadratic Hamiltonians. The Gaussian wave packets (2) have four integration constants: two constants for the classical position and momentum, and two variances for the position and momentum. The first two constants are equivalent to a complex parameter for the coherent state and the second two constants to the squeeze parameter and the squeeze angle. The joint entropy (8) is expressed entirely in terms of the uncertainty relation, which is independent of the centroid but depends on the position and momentum variances. We computed the joint entropy for a free particle, a harmonic oscillator and the Caldirola-Kanai Hamiltonian with an exponentially varying mass. The joint entropy depends only on the squeeze parameter and the squeeze angle as shown in Eqs. (13) for the free particle and (18) for the harmonic oscillator.

The general Gaussian wave packets have the initial entropy (14) that depends on the squeeze parameter and the squeeze angle. It is shown that the Leipnik's joint entropy satisfies the lower bound $S(t) \geq \ln(e/2)$ and monotonously increases for the range of squeeze angle $\pi \geq \vartheta \geq 0$, while for the remaining range $2\pi > \vartheta > \pi$ it first decreases until the lower bound, $\ln(e/2)$, is reached and then it monotonously increases. This counter-example implies that the joint entropy monotonously increases only for the initially maximally classical states that have the minimum uncertainty. Similarly, the joint entropy of the squeezed vacuum states of the initially maximally classical state for the harmonic oscillator oscillates between the maximum and the lower bound, $\ln(e/2)$.

In this paper, we have proposed that the random-phase entropy with respect to the squeeze angle leads to monotonously increasing entropy for the general Gaussian wave packets, which is shown explicitly for a free particle, a harmonic oscillator and the Caldirola-Kanai Hamiltonian. For a quadratic Hamiltonian with time-dependent mass or frequency, the joint entropy satisfies the lower bound (26) and the upper bound (27). It remains an open question whether the random-phase entropy monotonously increases for all time-dependent quadratic Hamiltonians.

Acknowledgments

SPK appreciates APCTP for the warm hospitality, where this paper was finished. This work has been supported by KORDI research program (PG47100) through KRCF(Korea Research Council of Fundamental Science and Technology) of Ministry of Education, Science and Technology.

[1] J. Dunkel and S. A. Trigger, Phys. Rev. A **71**, 052102 (2005).
[2] P. Garbaczewski, Phys. Rev. A **72**, 056101 (2005); V. Majerník and T. Opatrný, J. Phys. A: Math. Gen. **29**, 2187 (1996).
[3] Ö. Özcan, E. Aktüurk, and R. Sever, Int. J. Theor. Phys. **47**, 3207 (2008); E. Aktüurk, Ö. Özcan, and R. Sever, Int. J. Mod. Phys. B **23**, 2449 (2009).
[4] S. P. Kim, J. Korean Phys. Soc. **49**, 464 (2006) [arXiv:quant-ph/0511073].
[5] P. Caldirola, Nuovo Cimento **18**, 393 (1941); **77B**, 241 (1983); E. Kanai, Prog. Theor. Phys. **3**, 440 (1948).
[6] S. P. Kim and D. N. Page, Phys. Rev. A **64**, 012104 (2001).
[7] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[8] J-Y. Ji, J. K. Kim, and S. P. Kim, Phys. Rev. A **51**, 4268 (1995); J.-Y. Ji, J. K. Kim, S. P. Kim, and K-S. Soh, Phys. Rev. A **52**, 3352 (1995).
[9] I. A. Malkin, V. I. Man'ko, and D. A. Trifnov, Phys. Rev. D **2**, 1371 (1970).
[10] J. K. Kim and S. P. Kim, J. Phys. A **32**, 2711 (1999).
[11] S. P. Kim and C. H. Lee, Phys. Rev. D **62**, 125020 (2000).
[12] M. M. Nieto and D. R. Truax, New J. Phys. **2**, 18 (2000).
[13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 1994), p 560, formula 4.225.
[14] S. P. Kim, J. Phys. A: Math. Gen. **36**, 12089 (2003).